UDC 539.3

EQUATIONS OF THE AXISYMMETRIC STATE OF STRESS AND STRAIN OF A NONSHALLOW SPHERICAL SHELL OF NONLINEAR ELASTIC MATERIAL UNDER LARGE DEFORMATIONS

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A sequential derivation is given of the fundamental axisymmetric strain equations of a spherical shell made from nonlinear elastic material under the assumption of smallness of the relative elongations as compared with unity and under arbitrary angles of rotation.

The most widely used versions of the boundary value problems for the investigation of nonshallow shells under large deformations were proposed in [1 – 5]. The general formulation of boundary value problems for the investigation of nonshallow shells under large deformations is found in [6, 7]. Only the geometric nonlinearity is taken into account in all these papers. Equations and boundary conditions for the axisymmetric deformation of a nonshallow spherical shell under finite displacements, taking account of a physical nonlinearity, are presented herein.

1. Let us consider a spherical shell of radius R, thickness h in spherical $r\theta\varphi$ coordinates. Let u, w be the components of the displacement vector (Fig. 1), which are functions of just r, θ in this case. The following relationships:

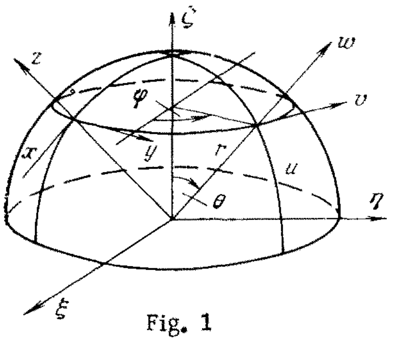

Fig. 1

$$\varepsilon_{rr} = \frac{\partial w}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \tag{1.1}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u}{\partial \theta} + w\right) + \frac{1}{2r^2}\left(\frac{\partial u}{\partial \theta} + w\right)^2 + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta} - u\right)^2 \tag{1.2}$$

$$\varepsilon_{\varphi\varphi} = \frac{1}{r\sin\theta}(w\sin\theta + u\cos\theta) + \frac{1}{2r^2\sin^2\theta}(w\sin\theta + u\cos\theta)^2 \tag{1.3}$$

$$\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial w}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\frac{1}{r} \tag{1.4}$$

hold for all the finite strain components different from zero ε_{rr}, $\varepsilon_{\theta\theta}$, $\varepsilon_{r\theta}$, $\varepsilon_{\varphi\varphi}$. The angle of rotation of the normal to the middle surface $r = R$ is given by

$$\omega = \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{u}{r} \tag{1.5}$$

Let us simplify the relationships (1.1) – (1.4) by assuming that the relative elongations and shears are negligibly small compared to unity. Then we obtain from (1.1) that $\partial w / \partial r \sim \varepsilon_{rr}$, and therefore, is also negligibly small compared to unity. On this basis, the second member in (1.1) can be neglected. We conclude analogously that the second member in (1.3) can also be discarded. Furthermore, since it has been established that $\partial w / \partial r \ll 1$, the third member in (1.4) can be neglected in comparison with the first two.

Therefore, under the assumption that the relative elongations and shears are considerably smaller than unity, the simplified finite strain components are

$$\varepsilon_{rr} = \frac{\partial w}{\partial r}, \qquad \varepsilon_{\varphi\varphi} = \frac{1}{r \sin\theta}(w \sin\theta + u\cos\theta) \tag{1.6}$$

$$\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial u}{\partial r} - \frac{u}{r} \tag{1.7}$$

2. Let us examine a thin shell; then the Kirchhoff's hypotheses are valid, as formulated in the following form:

$$u = u_0 + zu_1, \qquad w = w_0, \qquad z = r - R \tag{2.1}$$

$$\varepsilon_{r\theta} = 0, \qquad \sigma_r = 0 \tag{2.2}$$

The u_0, u_1, w_0 in (2.1) depend only on θ, and R is the radius of the shell middle surface. Substituting (2.1) into (1.7) and taking account of (2.2), we obtain for $|z| \ll R$

$$u_1 = \frac{1}{R}\left(u_0 - \frac{\partial w}{\partial \theta}\right) \tag{2.3}$$

Taking account of (2.3), let us write the formulas for the displacement and strain components as

$$u = u_0 - \frac{z}{R}\frac{\partial w}{\partial \theta}, \qquad w = w_0 \tag{2.4}$$

$$\varepsilon_{rr} = \frac{\partial w_0}{\partial r} = 0, \qquad \varepsilon_{\varphi\varphi} = \varepsilon_{\varphi\varphi}^{(0)} + z\varepsilon_{\varphi\varphi}^{(1)}, \qquad \varepsilon_{\theta\theta} = \varepsilon_{\theta\theta}^{(0)} + z\varepsilon_{\theta\theta}^{(1)} \tag{2.5}$$

Here

$$\varepsilon_{\varphi\varphi}^{(0)} = \frac{1}{R\sin\theta}(w_0 \sin\theta + u_0\cos\theta), \qquad \varepsilon_{\varphi\varphi}^{(1)} = -\frac{\operatorname{ctg}\theta}{R^2}\frac{\partial w_0}{\partial \theta} \tag{2.6}$$

$$\varepsilon_{\theta\theta}^{(0)} = \frac{1}{R}\left(\frac{\partial u_0}{\partial \theta} + w_0\right) + \frac{1}{2R^2}\left(\frac{\partial w_0}{\partial \theta} - u_0\right)^2 + \frac{1}{2R^2}\left(\frac{\partial u_0}{\partial \theta} + w_0\right)^2$$

$$\varepsilon_{\theta\theta}^{(1)} = -\frac{1}{R^2}\left[\frac{\partial^2 w_0}{\partial \theta^2} + \frac{1}{R}\left(\frac{\partial u_0}{\partial \theta} + w_0\right)\frac{\partial^2 w_0}{\partial \theta^2}\right]$$

3. Let us assume that the shell material is nonlinearly elastic, and its mechanical properties are characterized by the function $\Pi\ (\varepsilon_{ij})$ which yields the potential energy density stored per unit volume if the components of the finite strain tensor take the given values ε_{ij}. This function evidently characterizes the mechanical properties of the shell material completely.

Using the known Castigliano relations, as well as the second relation in (2.2), we obtain

$$\sigma_{rr} = \frac{\partial \Pi}{\partial \varepsilon_{rr}} = 0 \tag{3.1}$$

Let us express ε_{rr} from (3.1) in the terms of the remaining components of the finite strain tensor and let us substitute this expression in Π again. We then obtain a function

Π^* expressed in terms of the remaining five components of the finite strain tensor. Furthermore, let us assume that expressions for the components of the finite strain tensor, simplified in conformity with unity and the Kirchhoff's hypotheses, have been substituted into Π . We finally have

$$\Pi^* = \Pi^* [\varepsilon_{\varphi\varphi}^{(0)} + z\varepsilon_{\varphi\varphi}^{(1)},\ \varepsilon_{\theta\theta}^{(0)} + z\varepsilon_{\theta\theta}^{(1)}] \tag{3.2}$$

It is important for the sequel that we calculate the quantity W^*, the potential energy stored in the whole shell volume during its deformation. Under the assumptions made we have

$$W^* = \int\limits_{(V)} \Pi^* r^2 \sin\theta dr d\theta d\varphi = 2\pi R^2 \int\limits_{(s)} \Pi^* \left(1 + \frac{z}{R}\right)^2 \sin\theta dz d\theta \tag{3.3}$$

Let us introduce the quantity W defined by the relationship

$$W = \int\limits_{-h/2}^{h/2} \Pi^* \left(1 + \frac{z}{R}\right)^2 dz \approx \int\limits_{-h/2}^{h/2} \Pi^* dz \tag{3.4}$$

Evidently W depends on $\varepsilon_{\varphi\varphi}^{(0)}$, $\varepsilon_{\varphi\varphi}^{(1)}$, $\varepsilon_{\theta\theta}^{(0)}$, $\varepsilon_{\theta\theta}^{(1)}$. Therefore, we finally obtain for the internal shell strain energy

$$W^* = 2\pi R^2 \int\limits_0^{\theta_0} W \sin\theta d\theta \tag{3.5}$$

4. Keeping in mind the use of the principle of virtual displacements to derive the shell equilibrium equations, let us evaluate the variation of W^* upon variation of the displacements u_0, w_0. We have

$$\delta W^* = 2\pi R^2 \delta \int\limits_0^{\theta_0} W \sin\theta d\theta = 2\pi R^2 \int\limits_0^{\theta_0} [T_1 \delta\varepsilon_{\varphi\varphi}^{(0)} + T_2 \delta\varepsilon_{\theta\theta}^{(0)} + M_2 \delta\varepsilon_{\varphi\varphi}^{(1)} + M_1 \delta\varepsilon_{\theta\theta}^{(1)}] \sin\theta d\theta \tag{4.1}$$

$$T_1 = \frac{\partial W}{\partial \varepsilon_{\theta\theta}^{(0)}},\quad T_2 = \frac{\partial W}{\partial \varepsilon_{\varphi\varphi}^{(0)}},\quad M_1 = \frac{\partial W}{\partial \varepsilon_{\theta\theta}^{(1)}},\quad M_2 = \frac{\partial W}{\partial \varepsilon_{\varphi\varphi}^{(1)}} \tag{4.2}$$

If the relationships (2.5), (2.6) are taken into account and (4.1) is integrated by parts by interchanging the derivatives with u_0, w_0, we obtain, by taking into account that $M_1 = M_2$, $\varepsilon_{\theta\theta}^{(0)} = 0$ for $\theta = 0$

$$\begin{aligned}\delta W^* = 2\pi R^2 &\left\{\int\limits_0^{\theta_0} (T\delta u_0 + N\delta w_0)\, d\theta + \left[T_1 \frac{\sin\theta}{R} + T_1\left(\frac{\partial u_0}{\partial\theta} + w_0\right)\frac{\sin\theta}{R^2} - M_1 \frac{\partial^2 w_0}{\partial\theta^2}\frac{\sin\theta}{R^3}\right]_{\theta=\theta_0} \delta u_0 + \left[T_1\left(\frac{\partial w_0}{\partial\theta} - u_0\right)\frac{\sin\theta}{R^2} + \right.\right.\\ &\left.\frac{\partial}{\partial\theta}(M_1 \sin\theta)\frac{1}{R^2} - M_2\frac{\cos\theta}{R^2} + \frac{1}{R^3}\frac{\partial}{\partial\theta}\left(M_1 \sin\theta\left(\frac{\partial u_0}{\partial\theta} + w_0\right)\right)\right]_{\theta=\theta_0} \delta w_0 - \\ &\left[M_1\frac{\sin\theta}{R^2} + M_1\frac{\sin\theta}{R^3}\left(\frac{\partial u_0}{\partial\theta} + w_0\right)\right]_{\theta=\theta_0} \delta\frac{\partial w_0}{\partial\theta}\end{aligned} \tag{4.3}$$

Here T and N are given by the following relationships

$$\begin{aligned}T = T_2\frac{\cos\theta}{R} - \frac{1}{R}\frac{\partial}{\partial\theta}(T_1 \sin\theta) - T_1\left(\frac{\partial w_0}{\partial\theta} - u_0\right)\frac{\sin\theta}{R^2} - \\ \frac{1}{R^2}\frac{\partial}{\partial\theta}\left[T_1 \sin\theta\left(\frac{\partial u_0}{\partial\theta} + w_0\right) + \frac{1}{R^3}\frac{\partial}{\partial\theta}\left(M_1 \sin\theta\frac{\partial^2 w_0}{\partial\theta^2}\right)\right]\end{aligned} \tag{4.4}$$

$$N = T_2 \frac{\sin\theta}{R} + T_1 \frac{\sin\theta}{R} - \frac{1}{R^2}\frac{\partial}{\partial\theta}\left[T_1 \sin\theta \left(\frac{\partial w_0}{\partial\theta} - u_0\right)\right] + \tag{4.5}$$
$$\frac{\sin\theta}{R^2} T_1 \left(\frac{\partial u_0}{\partial\theta} + w_0\right) + \frac{1}{R^2}\frac{\partial}{\partial\theta}(M_2 \cos\theta) - \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}(M_1 \sin\theta) -$$
$$\frac{1}{R^3} M_1 \frac{\partial^2 w_0}{\partial\theta^2} \sin\theta - \frac{1}{R^3}\frac{\partial^2}{\partial\theta^2}\left[M_1 \sin\theta \left(\frac{\partial u_0}{\partial\theta} + w_0\right)\right]$$

5. Let us turn to the computation of the elementary work of the external forces applied to a shell in possible displacements. Let us assume that a system of stresses q_r^+, q_θ^+ is applied to the bounding sphere of the shell $z = h/2$, and the system q_r^-, q_θ^- to the bounding sphere $z = -h/2$. The mentioned loads can depend not only on the coordinate θ, but also on the displacements u, w. In the case under consideration the elementary work δA_1 of these stresses resultants in possible displacements is expressed by

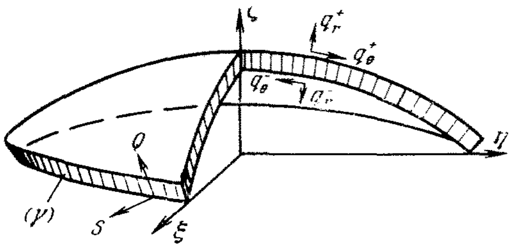

Fig. 2

$$\delta A_1 = 2\pi R^2 \int_0^{\theta_0} \left\{\left[q_\theta^+ \delta u\left(+\frac{h}{2}\right) + q_r^+ \delta w\left(+\frac{h}{2}\right)\right]\left(1 + \frac{h}{2R}\right) + \right. \tag{5.1}$$
$$\left.\left[q_\theta^- \delta u\left(-\frac{h}{2}\right) + q_r^- \delta w\left(-\frac{h}{2}\right)\right]\left(1 - \frac{h}{2R}\right)\right\} \sin\theta d\theta$$

Let us substitute the relations for δu and δw from (2.4) into (5.1), discarding insignificant terms of order h/R, we have

$$\delta A_1 = 2\pi R^2 \int_0^{\theta_0} \left(q_\theta \delta u_0 + q_r \delta w_0 + q_r^* \delta \frac{\partial w_0}{\partial\theta}\right) \sin\theta d\theta \tag{5.2}$$

$$q_\theta = q_\theta^+ + q_\theta^-, \quad q_r = q_r^+ + q_r^-, \quad q_r^* = \frac{h}{2R}(q_\theta^+ - q_\theta^-)$$

We finally obtain the following expression for δA_1 from (5.2)

$$\delta A_1 = 2\pi R^2 \int_0^{\theta_0} \left\{q_\theta \delta u_0 + \left[q_r \frac{\partial q_r^*}{\partial\theta} - q_r^* \operatorname{ctg}\theta\right]\delta w_0\right\} \sin\theta d\theta + \tag{5.3}$$
$$2\pi R^2 q_r^* \sin\theta_0 \delta w_0$$

We have

$$\delta A_2 = \int_{(\gamma)} [Q(r)\delta w + S(r)\delta u]\, r dr\, d\varphi \tag{5.4}$$

where γ is the endface surface (Fig. 2). Substituting (2.4) into (5.4) and neglecting terms of order h/R whereever possible, we obtain

$$\delta A_2 = 2\pi R \int_{-h/2}^{h/2} \left[Q(r)\delta w_0 S(r)\delta u_0 - S(r)\frac{z}{R}\delta\frac{\partial w_0}{\partial\theta}\right] dz = \tag{5.5}$$
$$2\pi R \left[Q^* \delta w_0 + S^* \delta u_0 - S^{**} \delta \frac{\partial w_0}{\partial\theta}\right]_{\theta=\theta_0}$$

$$Q^* = \int_{-h/2}^{h/2} Q(r)\,dz, \qquad S^* = \int_{-h/2}^{h/2} S(r)\,dz, \qquad S^{**} = \int_{-h/2}^{h/2} S(r)\,\frac{z\,dz}{R}$$

6. Under the assumptions made let us write the Lagrange principle of possible virtual displacements

$$\delta W^* - \delta A_1 - \delta A_2 = 0 \tag{6.1}$$

In conformity with (4.5), (5.3) and (5.5), by considering the variations δu_0 and δw_0 to be independent, we obtain the following equations and boundary conditions from (6.1)

$$\begin{aligned}
&\frac{\partial T_1}{\partial\theta} + (T_1 - T_2)\operatorname{ctg}\theta + \frac{1}{R}\frac{\partial T_1}{\partial\theta}\left(\frac{\partial u_0}{\partial\theta} + w_0\right) + T_1\omega + \\
&\quad \frac{1}{R}T_1\left(\frac{\partial^2 u_0}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial u_0}{\partial\theta} + \frac{\partial w_0}{\partial\theta} + w_0\operatorname{ctg}\theta\right) - \frac{1}{R^2}\frac{\partial M_1}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2} - \\
&\quad \frac{1}{R^2}M_1\left(\frac{\partial^3 w_0}{\partial\theta^3} + \operatorname{ctg}\theta\frac{\partial^2 w_0}{\partial\theta^2}\right) + q_0 R = 0 \\
&\frac{\partial^2 M_1}{\partial\theta^2} + \left(\frac{\partial M_1}{\partial\theta} - \frac{\partial M_2}{\partial\theta}\right)\operatorname{ctg}\theta + \frac{\partial M_1}{\partial\theta}\operatorname{ctg}\theta - (M_1 - M_2) + \\
&\quad \frac{1}{R}\frac{\partial^2 M_1}{\partial\theta^2}\left(\frac{\partial u_0}{\partial\theta} + w_0\right) + \frac{2}{R}\frac{\partial M_1}{\partial\theta}\left(\frac{\partial^2 u_0}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial u_0}{\partial\theta} + \frac{\partial w_0}{\partial\theta} + \operatorname{ctg}\theta\, w_0\right) + \\
&\quad \frac{1}{R}M_1\left(\frac{\partial^3 u_0}{\partial\theta^3} + \operatorname{ctg}\theta\frac{\partial^2 u_0}{\partial\theta^2} - \frac{\partial u_0}{\partial\theta} + 2\frac{\partial^2 w_0}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial w_0}{\partial\theta} - w_0\right) - \\
&\quad R(T_1 + T_2) + \frac{\partial T_1}{\partial\theta}\left(\frac{\partial w_0}{\partial\theta} - u_0\right) - T_1\left(2\frac{\partial u_0}{\partial\theta} + u_0\operatorname{ctg}\theta - \frac{\partial^2 w_0}{\partial\theta^2} - \right. \\
&\quad \left.\operatorname{ctg}\theta\frac{\partial w_0}{\partial\theta} + w_0\right) + \left(q_r - \frac{\partial q_r^*}{\partial\theta}\right)R^2 - q_r^*\operatorname{ctg}\theta R^2 = 0 \\
&\left\{\left[T_1 + T_1\left(\frac{\partial u_0}{\partial\theta} + w_0\right)\frac{1}{R} - M_1\frac{\partial^2 w_0}{\partial\theta^2}\frac{1}{R^2}\right]\sin\theta - S^*\right\}_{\theta=\theta_0} = 0 \\
&\left\{T_1 R\omega\sin\theta + \frac{\partial}{\partial\theta}(M_1\sin\theta) - M_2\cos\theta + \frac{1}{R}\frac{\partial}{\partial\theta}\left[M_1\sin\theta\left(\frac{\partial u_0}{\partial\theta} + w_0\right)\right] - \right. \\
&\quad \left. RQ^* + q_r^* R^2\sin\theta\right\}_{\theta=\theta_0} = 0 \\
&\left[M_1\sin\theta + M_1\sin\theta\left(\frac{\partial u_0}{\partial\theta} + w_0\right) + RS^{**}\right|_{\theta=\theta_0} = 0
\end{aligned} \tag{6.2}$$

The second equation in (6.2) can be simplified if the expression $T_1\operatorname{ctg}\theta + \partial T_1/\partial\theta$ is determined from the first and substituted into the second. We then obtain

$$\begin{aligned}
&\frac{\partial^2 M_1}{\partial\theta^2} + \frac{\partial(M_1 - M_2)}{\partial\theta}\operatorname{ctg}\theta + \frac{\partial M_1}{\partial\theta}\operatorname{ctg}\theta - (M_1 - M_2) + \\
&\quad \frac{1}{R}\frac{\partial^2 M_1}{\partial\theta^2}\left(\frac{\partial u_0}{\partial\theta} + w_0\right) + \frac{2}{R}\frac{\partial M_1}{\partial\theta}\left(\frac{\partial^2 u_0}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial u_0}{\partial\theta} + \frac{\partial w_0}{\partial\theta} + w_0\operatorname{ctg}\theta\right) + \\
&\quad \frac{1}{R}M_1\left(\frac{\partial^3 u_0}{\partial\theta^3} + \operatorname{ctg}\theta\frac{\partial^2 u_0}{\partial\theta^2} - \frac{\partial u_0}{\partial\theta} + 2\frac{\partial^2 w_0}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial w_0}{\partial\theta} - w_0\right) - \\
&\quad R(T_1 + T_2) + \omega\left[RT_2\operatorname{ctg}\theta - \frac{\partial T_1}{\partial\theta}\left(\frac{\partial u_0}{\partial\theta} + w_0\right) - RT_1\omega - \right. \\
&\quad T_1\left(\frac{\partial^2 u_0}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial u_0}{\partial\theta} + \frac{\partial w_0}{\partial\theta} + w_0\operatorname{ctg}\theta\right) + \frac{1}{R}\frac{\partial M_1}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2} + \\
&\quad \left. M_1\left(\frac{\partial^3 w_0}{\partial\theta^3} + \operatorname{ctg}\theta\frac{\partial^2 w_0}{\partial\theta^2}\right) - q_0 R\right] - T_1\left(2\frac{\partial u_0}{\partial\theta} - \frac{\partial^2 w_0}{\partial\theta^2} + w_0\right) + \\
&\quad \left(q_r - \frac{\partial q_r^*}{\partial\theta}\right)R^2 - q_r^*\operatorname{ctg}\theta R^2 = 0
\end{aligned}$$

The relationship (4.2) should be added to (6.2). We then obtain a system of equations and boundary conditions to determine u_0, w_0, T_1, T_2, M_1, M_2. It is often convenient to integrate this system directly in the form presented here. In some cases it is expedient to eliminate the static factors by means of (4.2). Consequently, we obtain a system of two equations in u_0, w_0. It is expedient to use it if purely geometric conditions are given. On the basis of the boundary value problems obtained, the influence of the lack of shallowness and of the physical nonlinearity on the critical state of the shell can be investigated in particular.

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AN ENERGY INEQUALITY IN THE THEORY OF PLATE BENDING

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The construction of error estimates of approximate theories of plates is based on inequalities relating the three-dimensional elastic energy of the plate and the elastic energy according to a two-dimensional approximate theory.

The inequality

$$E_0(u_\alpha) \leqslant E(w, w_\alpha) \tag{1}$$

has been proved in [1] in the case of extension of an isotropic homogeneous linearly elastic plate of constant thickness h. Here E is the three-dimensional elastic energy, w_α are the tangential components of the displacement vector, w is the displacement along the normal to the plate (*), E_0 is the elastic energy by the theory of the plane

*) For the extension w_α is an even and w is an odd function of the transverse coordinate x, while for bending w_α is an odd, and w an even function of x.